

Indian Statistical Institute, Bangalore

M. Math.

Second Year, First Semester

Operator Algebra
Mid-term Examination
Maximum marks: 100

Instructor: B V Rajarama Bhat
Date : September 9, 2026
Time: 3 hours

- (1) Give a detailed proof of the fact that in a unital commutative Banach algebra the maximal ideals are in one to one correspondence with complex homomorphisms. [15]
- (2) Show that there is no complex homomorphism on the algebra $\mathcal{B}(l^2)$, where l^2 is the space of square summable sequences $l^2 = \{(a_n) : a_n \in \mathbb{C}, \sum_{n=1}^{\infty} |a_n|^2 < \infty\}$ with usual inner product. [15]
- (3) Prove the polynomial spectral mapping theorem in unital Banach algebras. [15]
- (4) Let G be a finite abelian group with n elements, where $n \geq 2$. Consider discrete topology on G . Let $\mathcal{A}$ be the space of continuous functions on G considered as a Banach algebra with respect to convolution product. Identify the Gelfand space Δ and the Gelfand map, $a \mapsto \hat{a}$ for $a \in \mathcal{A}$. [15]
- (5) Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{A}$ be a unital C^* -subalgebra of $\mathcal{B}(\mathcal{H})$. Then the *commutant* of $\mathcal{A}$ is denoted by $\mathcal{A}'$ and is defined by
$$\mathcal{A}' = \{X \in \mathcal{B}(\mathcal{H}) : XA = AX \text{ for all } A \in \mathcal{A}\}.$$
(i) Show that $\mathcal{A}'$ is a unital C^* -algebra. (ii) Show that the double commutant $(\mathcal{A}')'$ is a C^* -algebra containing $\mathcal{A}$. [15]
- (6) Let $\mathcal{A}$ be a unital C^* -algebra. Suppose $a, b \in \mathcal{A}$. (i) If a, b are commuting show that
$$\sigma(a+b) \subseteq \sigma(a) + \sigma(b),$$
$$\sigma(ab) \subseteq \sigma(a) \cdot \sigma(b),$$
where $\sigma(a) + \sigma(b)$ and $\sigma(a) \cdot \sigma(b)$, are defined as
$$\{\lambda + \mu : \lambda \in \sigma(a), \mu \in \sigma(b)\},$$
$$\{\lambda \cdot \mu : \lambda \in \sigma(a), \mu \in \sigma(b)\}.$$
(ii) Give an example to show that the containment in (i) could be proper, that is, equalities may not hold. [15]
- (7) Let $\mathcal{A}$ be the C^* algebra $C[0, 1]$ of continuous complex valued functions on $[0, 1]$ with supremum norm. Define $\phi : \mathcal{A} \rightarrow \mathbb{C}$ be the linear map $\phi(f) = \frac{2f(0)+f(1)}{3}$. Show that ϕ is a state on $C[0, 1]$. Verify that the norm of ϕ is attained at the identity. [15]